On the Minimum Mean p-th Error in Gaussian Noise Channels and Its Applications

Alex Dytso, Ronit Bustin, Daniela Tuninetti,
Natasha Devroye, H. Vincent Poor,
and Shlomo Shamai (Shitz)

UIC Department of Electrical and Computer Engineering
UNIVERSITY OF ILLINOIS AT CHICAGO
COLLEGE OF ENGINEERING

PRINCETON UNIVERSITY

TECHNION
Israel Institute of Technology

Outline

- Notation & Past Work
- max-MMSE Problem
- Minimum Mean p-th Error (MMPE)
- Properties of the MMPE
- Applications

Notation

$$\mathbf{X} \in \mathbb{R}^n, \ \frac{1}{n}\mathrm{Tr}\left(\mathbb{E}[\mathbf{X}\mathbf{X}^T]\right) \le 1,$$
$$\mathbf{Y}_{\mathrm{snr}} = \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z},$$
$$I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{n} I(\mathbf{X}; \mathbf{Y}_{\mathrm{snr}}),$$
$$\mathrm{mmse}(\mathbf{X}|\mathbf{Y}_{\mathrm{snr}}) = \mathrm{mmse}(\mathbf{X}, \mathrm{snr}) := \frac{1}{n}\mathrm{Tr}\left(\mathbb{E}\left[\mathrm{cov}(\mathbf{X}|\mathbf{Y}_{\mathrm{snr}})\right]\right)$$

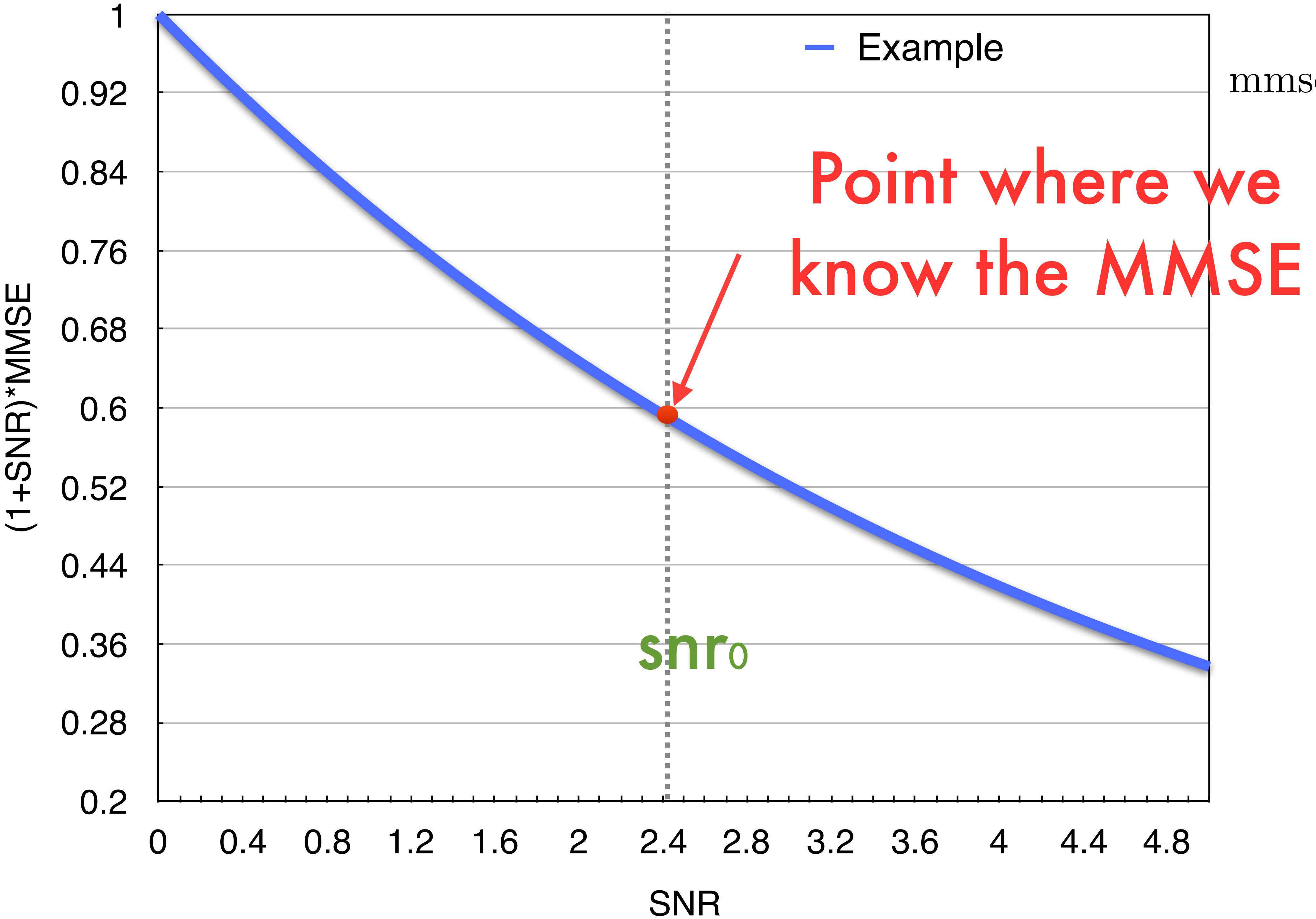
The I-MMSE relationship

$$\frac{d}{d\mathrm{snr}} I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{2}\mathrm{mmse}(\mathbf{X}, \mathrm{snr})$$

$$I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}, t)dt$$

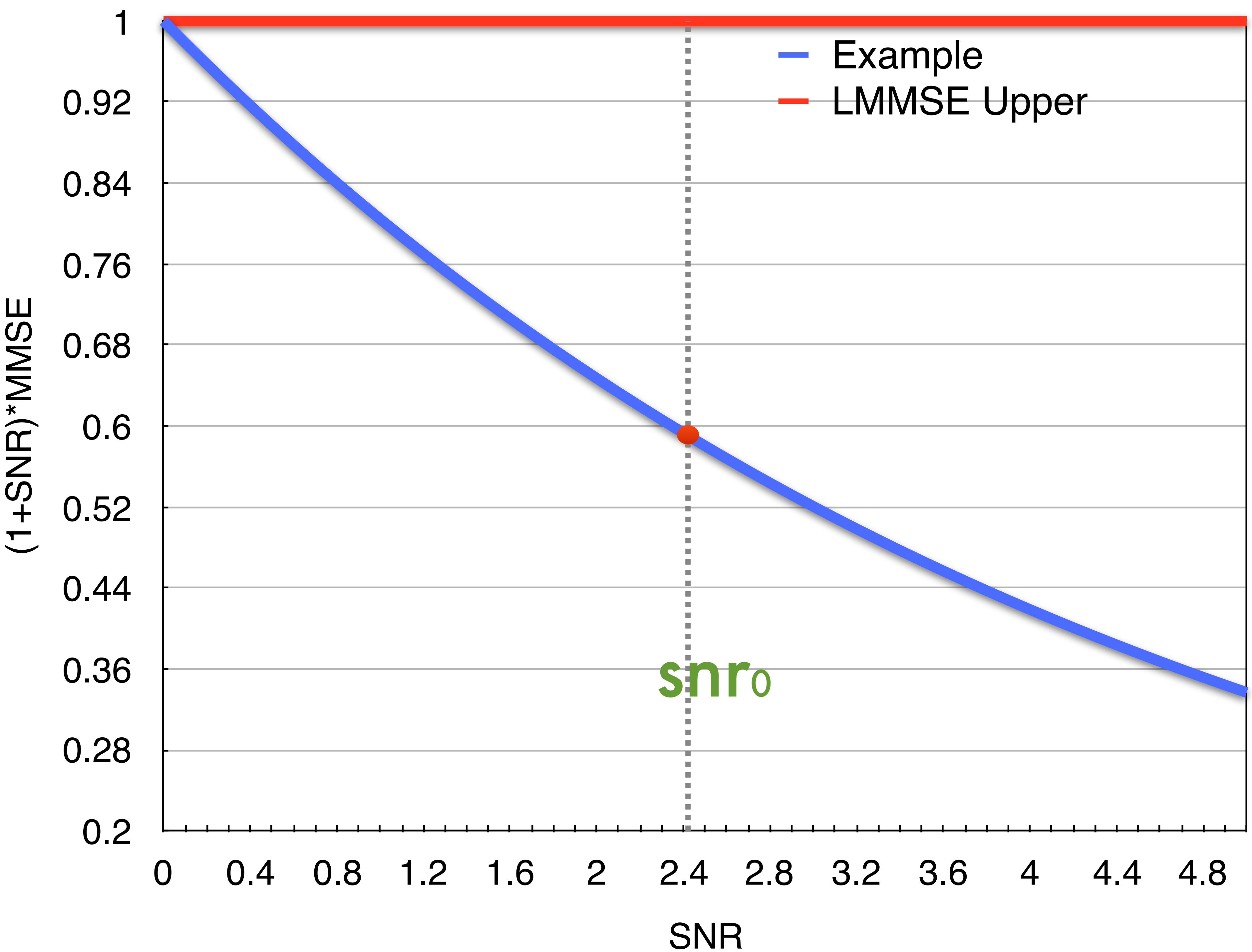
D. Guo, S. Shamai, and S. Verdu´, "Mutual information and minimum mean-square error in Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1261–1282, April 2005.

Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

Review: MMSE bounds



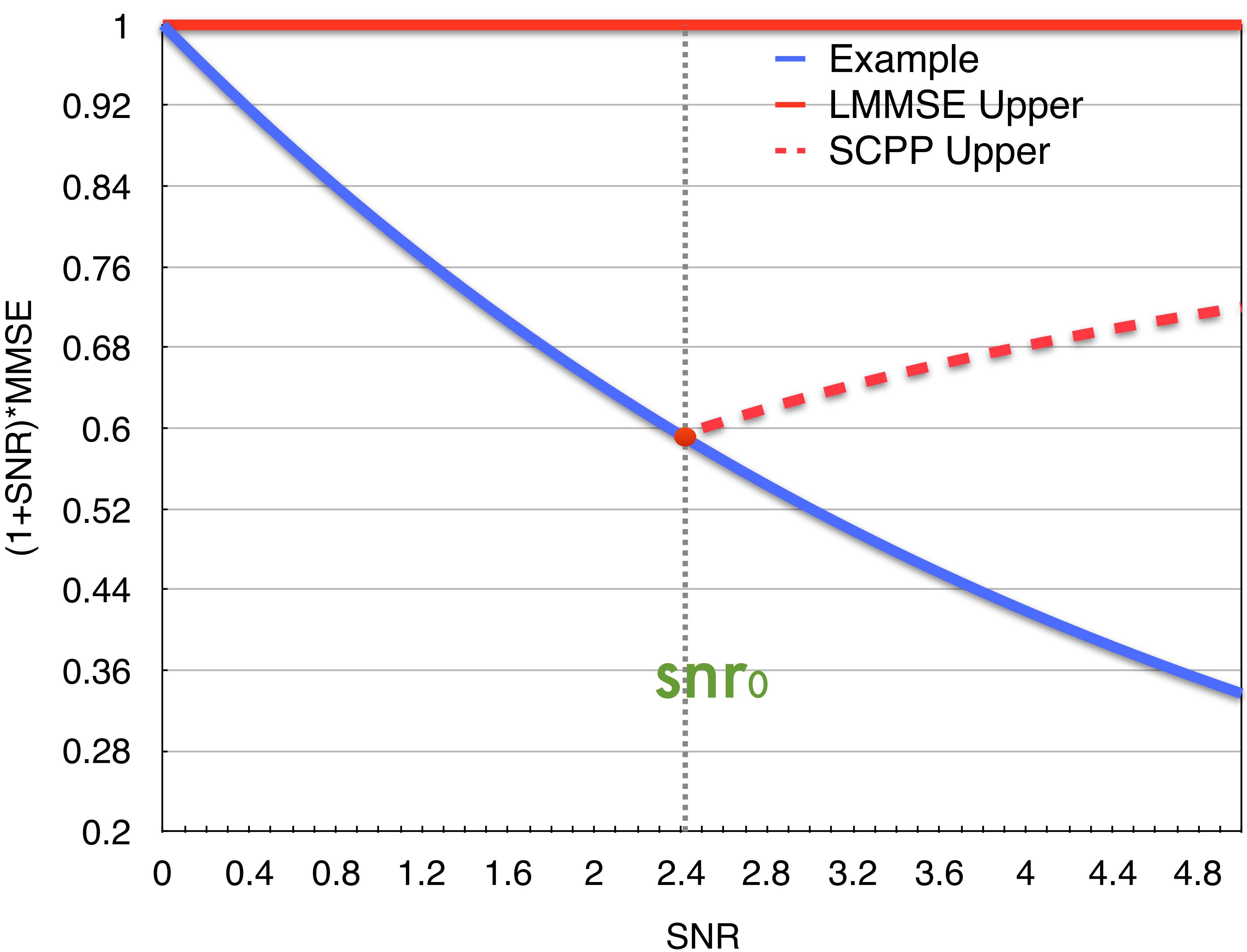
$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1+\beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1+\mathrm{snr}}$$

only power constraint

Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1+\beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1+\mathrm{snr}}$$

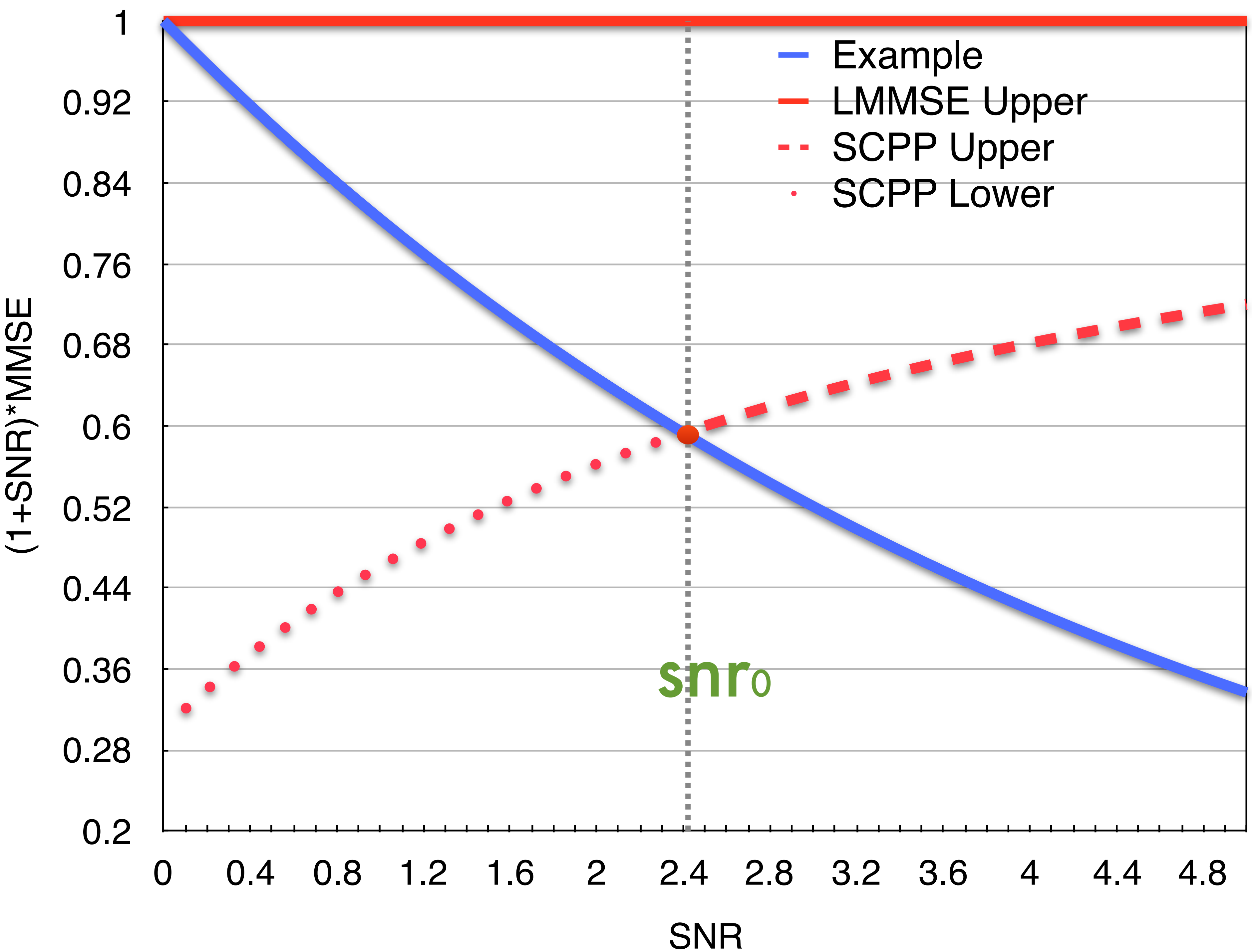
SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1+\beta \mathrm{snr}}$$

for $\mathrm{snr} \geq \mathrm{snr}_0$

MMSE constraint at snr_0

MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1+\beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1+\mathrm{snr}}$$

SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1+\beta \mathrm{snr}}$$

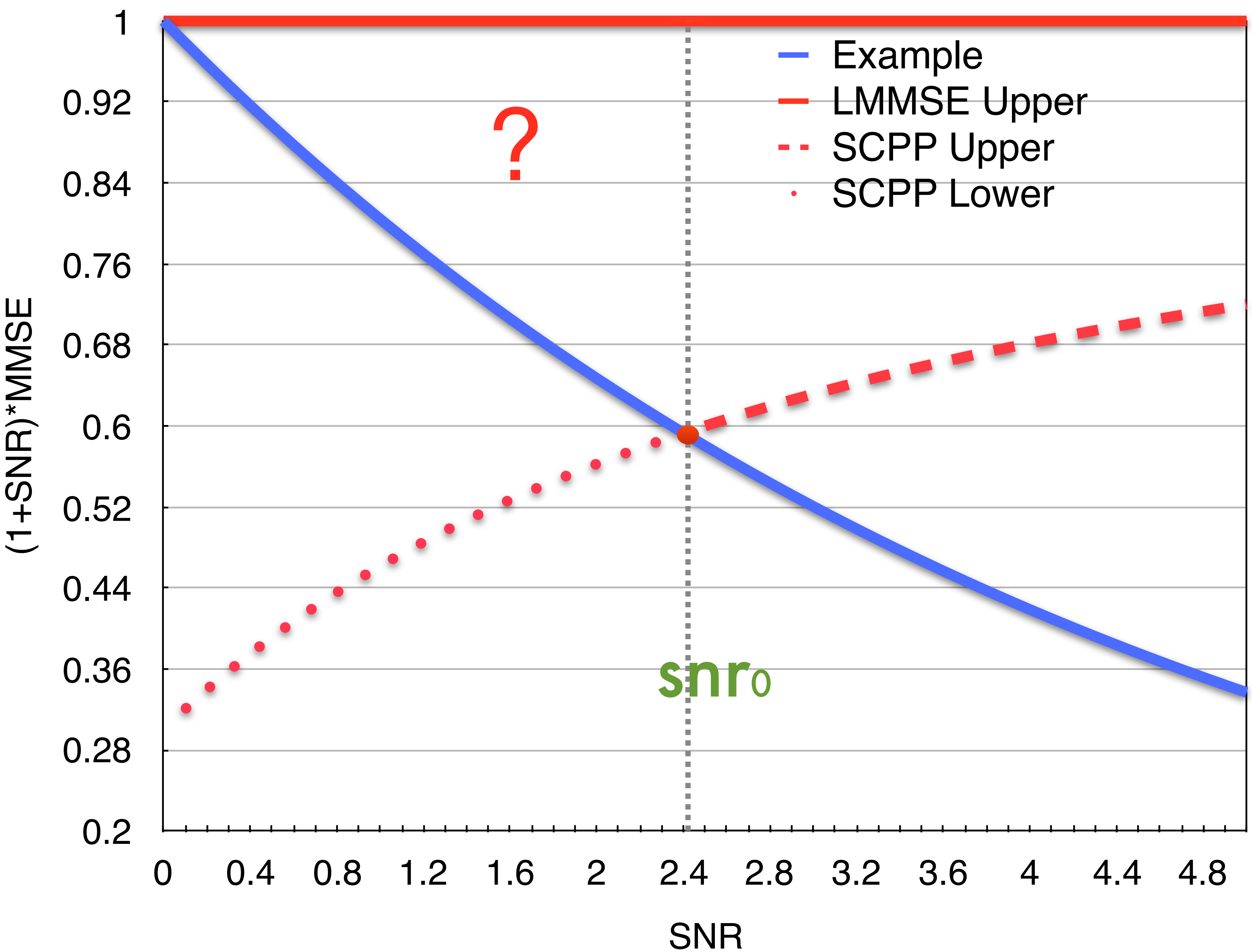
for $\mathrm{snr} \geq \mathrm{snr}_0$

SCPP l.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \geq \frac{\beta}{1+\beta \mathrm{snr}}$$

for $\mathrm{snr} \leq \mathrm{snr}_0$

Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1 + \mathrm{snr}}$$

SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \geq \mathrm{snr}_0$

SCPP l.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \geq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \leq \mathrm{snr}_0$

ISIT 2016 BARCELONA

MMSE Bounds: Single Crossing Point Property

(SCPP.) For any fixed $\mathbf{X}$, suppose that $\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1+\beta \mathrm{snr}_0}$, for some fixed $\beta \geq 0$. Then

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1+\beta \mathrm{snr}}, \quad \mathrm{snr} \in [\mathrm{snr}_0, \infty)$$
$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \geq \frac{\beta}{1+\beta \mathrm{snr}}, \quad \mathrm{snr} \in [0, \mathrm{snr}_0).$$

SCPP+I-MMSE important tool for derive converses:

- Version of EPI
- BC
- Wiretap
- MIMO channels

D. Guo, Y. Wu, S. Shamai, and S. Verdu, "Estimation in Gaussian noise: Properties of the minimum mean-square error," IEEE Trans. Inf. Theory, vol. 57, no. 4, pp. 2371–2385, April 2011.

R. Bustin, R. Schaefer, H. Poor, and S. Shamai, "On MMSE properties of optimal codes for the Gaussian wiretap channel," in Proc. IEEE Inf. Theory Workshop, April 2015, pp. 1–5.

R. Bustin, M. Payaro, D. Palomar, and S. Shamai, "On MMSE crossing properties and implications in parallel vector Gaussian channels," IEEE Trans. Inf. Theory, vol. 59, no. 2, pp. 818–844, Feb 2013.

max-MMSE problem

Motivated by the need of a complementary (to the SCPP) upper bound for MMSE

max-MMSE problem

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) := \sup_{\mathbf{X}} \mathrm{mmse}(\mathbf{X}, \mathrm{snr}),$$

$$\text{s.t. } \frac{1}{n}\mathrm{Tr}\left(\mathbb{E}[\mathbf{X}\mathbf{X}^{\mathrm{T}}]\right) \leq 1,$$

$$\text{and } \mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) \leq \frac{\beta}{1+\beta\mathrm{snr}_0}.$$

From SCPP

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1+\beta\mathrm{snr}}$$

$$\text{for snr} \geq \mathrm{snr}_0$$

solution for snr>snr$_0$: Gaussian input with reduced power

In the rest focus on snr<snr$_0$

Bounds on max-MMSE

$$\mathrm{M}_\infty(\mathrm{snr},\mathrm{snr}_0,\beta)=\begin{cases}\frac{1}{1+\mathrm{snr}}, & \mathrm{snr}<\mathrm{snr}_0,\\ \frac{\beta}{1+\beta\mathrm{snr}}, & \mathrm{snr}\geq\mathrm{snr}_0,\end{cases}.$$

- Converse: LMMSE+SCPP.
- Achievability: superposition coding with Gaussian codebooks.

Discontinuity, or *phase transition*, at $\mathrm{snr}=\mathrm{snr}_0$

N. Merhav, D. Guo, and S. Shamai, “Statistical physics of signal estimation in Gaussian noise: Theory and examples of phase transitions,” *IEEE Trans. Inf. Theory*, vol. 56, no. 3, pp. 1400–1416, March 2010.

Bounds on max-MMSE

$$\mathrm{M}_\infty(\mathrm{snr}, \mathrm{snr}_0, \beta) = \begin{cases} \frac{1}{1+\mathrm{snr}}, & \mathrm{snr} < \mathrm{snr}_0, \\ \frac{\beta}{1+\beta \mathrm{snr}}, & \mathrm{snr} \geq \mathrm{snr}_0, \end{cases}.$$

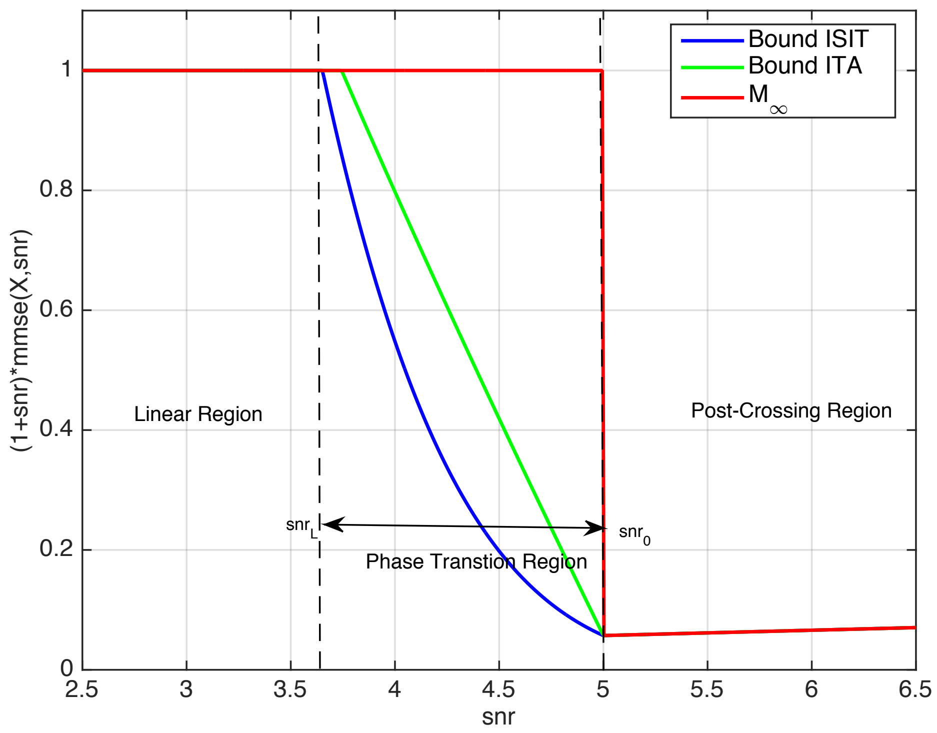
Discontinuity, or *phase transition*, at snr=snr$_0$

For Finite *n*

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) = \begin{cases} \frac{1}{1+\mathrm{snr}}, & \mathrm{snr} \leq \mathrm{snr}_L, \\ T_n(\mathrm{snr}, \mathrm{snr}_0, \beta), & \mathrm{snr}_L \leq \mathrm{snr} \leq \mathrm{snr}_0, \\ \frac{\beta}{1+\beta \mathrm{snr}}, & \mathrm{snr}_0 \leq \mathrm{snr}, \end{cases}$$

***phase transition* region**

for some snr_L.



ISIT 2016 BARCELONA

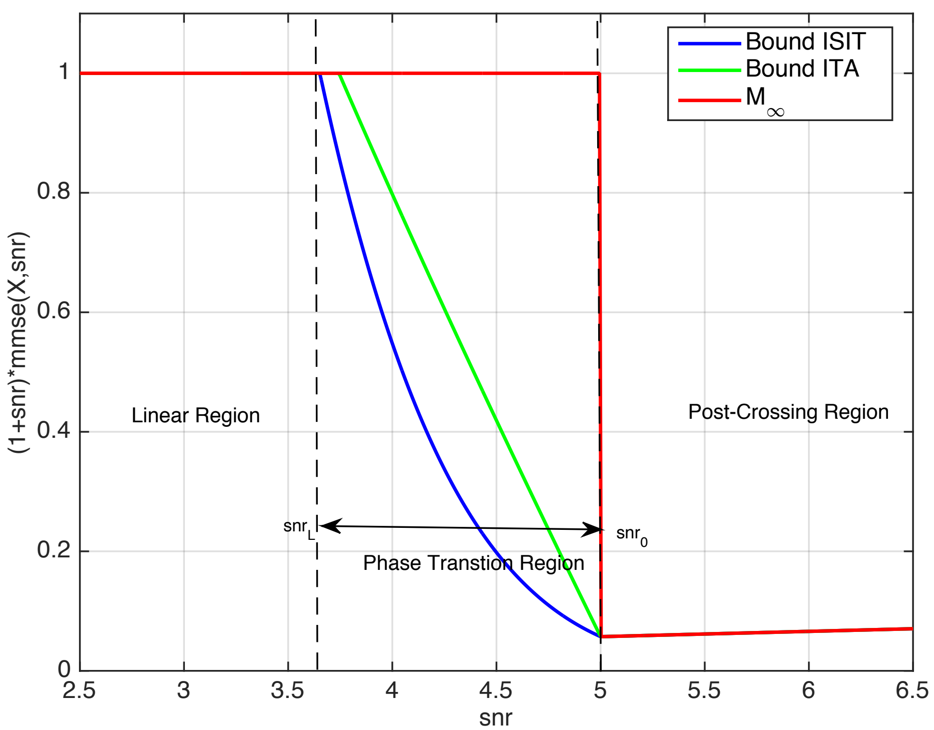
Bounds on max-MMSE

For Finite *n*

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) = \begin{cases} \frac{1}{1+\mathrm{snr}}, & \mathrm{snr} \le \mathrm{snr}_L, \\ T_n(\mathrm{snr}, \mathrm{snr}_0, \beta), & \mathrm{snr}_L \le \mathrm{snr} \le \mathrm{snr}_0, \\ \frac{\beta}{1+\beta \mathrm{snr}}, & \mathrm{snr}_0 \le \mathrm{snr}, \end{cases}$$

phase transition region

for some snr_L.



Theorem.

$$T_n(\mathrm{snr}, \mathrm{snr}_0, \beta) \le \frac{\beta}{1+\beta \mathrm{snr}_0} + \kappa_n \left(\frac{1}{\mathrm{snr}} - \frac{1}{\mathrm{snr}_0} \right),$$

where $\kappa_n \le n+2$.

The width of the phase transition region is $W(n) = O\left(n^{-1}\right)$.

A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, S. Shamai, and H. V. Poor, "New bounds on MMSE and applications to communication with the disturbance constraint," Submitted to *IEEE Trans. Inf. Theory,* https://arxiv.org/pdf/1603.07628, 2016.

ISIT 2016 BARCELONA

MMPE

The p-norm of a random vector $\mathbf{U} \in \mathbb{R}^n$ is give by

$$\|\mathbf{U}\|_p^p = \frac{1}{n}\mathrm{E}\left[\mathrm{Tr}^{\frac{p}{2}}\left(\mathbf{U}\mathbf{U}^T\right)\right].$$

For $n = 1$, $\|U\|_p = \mathrm{E}[|U|^p]$.

Example: $\mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$

$$n\|\mathbf{U}\|_p^p = 2^{\frac{p}{2}} \frac{\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{n}{2}\right)},$$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

MMPE

We define the minimum mean p-th error (MMPE) of estimating $\mathbf{X}$ from $\mathbf{Y}$ as

$$\boxed{\mathrm{mmpe}(\mathbf{X}|\mathbf{Y};p) := \inf_f \|\mathbf{X}-f(\mathbf{Y})\|_p^p,}$$
$$:= \inf_f n^{-1}\mathbb{E}\left[\mathrm{Err}^{\frac{p}{2}}\left(\mathbf{X}, f(\mathbf{Y})\right)\right].$$

We donate the optimal estimator by $\boxed{f_p(\mathbf{X}|\mathbf{Y}).}$
Example: for $p=2$

$$\mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p=2) = \mathrm{mmse}(\mathbf{X}, \mathrm{snr}),$$
$$f_{p=2}(\mathbf{X}|\mathbf{Y}) = \mathbb{E}[\mathbf{X}|\mathbf{Y}].$$

Study Properties

Properties of the MMPE

Optimal Estimator

Theorem. For $\text{mmpe}(\mathbf{X}, \text{snr}, p)$ and $p \geq 0$ the optimal estimator is given by the following point-wise relationship:

$$f_p(\mathbf{X}|\mathbf{Y}=\mathbf{y}) = \arg\min_{\mathbf{v}\in\mathbb{R}^n} \mathrm{E}\left[\mathsf{Err}^{\frac{p}{2}}(\mathbf{X},\mathbf{v})|\mathbf{Y}=\mathbf{y}\right].$$

Orthogonality `Like' Property

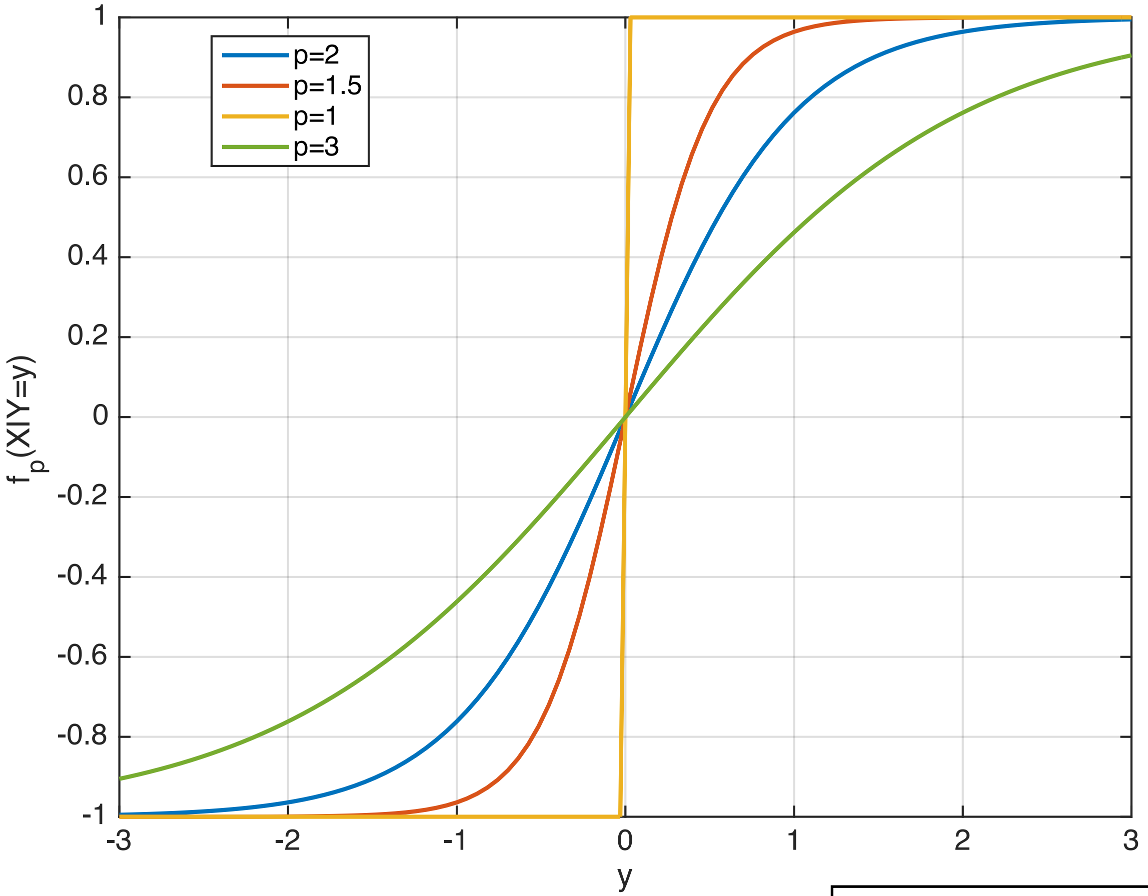
Theorem. For any $1 \leq p < \infty$ optimal estimator $f_p(\mathbf{X}|\mathbf{Y})$ satisfies

$$\mathrm{E}\left[\left(\mathbf{W}^T\mathbf{W}\right)^{\frac{p-2}{2}} \cdot \mathbf{W}^T \cdot g(\mathbf{Y})\right] = 0,$$

where $\mathbf{W} = \mathbf{X} - f_p(\mathbf{X}|\mathbf{Y})$ for any deterministic function $g(\cdot)$.

Examples of Optimal Estimators

- if $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ then for $p \geq 1$ $f_p(\mathbf{X}|\mathbf{Y}=\mathbf{y}) = \frac{\sqrt{\text{snr}}}{1+\text{snr}}\mathbf{y}$ $\boxed{\text{(i.e., linear);}}$
- if $X = \{\pm 1\}$ equally likely (BPSK) then for $p \geq 1$ $f_p(X|Y) = \tanh\left(\frac{\sqrt{\text{snr}}}{p-1} y\right)$.



S. Sherman, “Non-mean-square error criteria,” *IRE Transactions on Information Theory*, vol. 3, no. 4, pp. 125–126, 1958.

Properties of Optimal Estimators

E[X|Y]

1. if $0 \leq X \in \mathbb{R}^1$ then $0 \leq \mathrm{E}[X|Y]$,
2. (Linearity) $\mathrm{E}[a\mathbf{X} + b|\mathbf{Y}] = a\mathrm{E}[\mathbf{X}|\mathbf{Y}] + b$,
3. (Stability) $\mathrm{E}[g(\mathbf{Y})|\mathbf{Y}] = g(\mathbf{Y})$ for any function $g(\cdot)$,
4. (Idempotent) $\mathrm{E}[\mathrm{E}[\mathbf{X}|\mathbf{Y}]|\mathbf{Y}] = \mathrm{E}[\mathbf{X}|\mathbf{Y}]$,
5. (Degradedness)0 $\mathrm{E}\left[\mathbf{X}|\mathbf{Y}_{\mathrm{snr}}, \mathbf{Y}_{\mathrm{snr}_0}\right] = \mathrm{E}\left[\mathbf{X}|\mathbf{Y}_{\mathrm{snr}_0}\right]$, for $\mathbf{X} \to \mathbf{Y}_{\mathrm{snr}_0} \to \mathbf{Y}_{\mathrm{snr}}$,
6. (Shift Invariance) $\mathrm{mmse}(\mathbf{X} + a, \mathrm{snr}) = \mathrm{mmse}(\mathbf{X}, \mathrm{snr})$,
7. (Scaling) $\mathrm{mmse}(a\mathbf{X}, \mathrm{snr}) = a^2 \mathrm{mmse}(\mathbf{X}, a^2\mathrm{snr})$.

$$\mathrm{E}\left[\mathrm{E}\left[X|Y\right]\right] = \mathrm{E}\left[X\right]$$

f_p(X|Y)

1. if $0 \leq X \in \mathbb{R}^1$ then $0 \leq f_p(X|Y)$,
2. (Linearity) $f_p(a\mathbf{X} + b|\mathbf{Y}) = a f_p(\mathbf{X}|\mathbf{Y}) + b$,
3. (Stability) $f_p(g(\mathbf{Y})|\mathbf{Y}) = g(\mathbf{Y})$ for any function $g(\cdot)$,
4. (Idempotent) $f_p(f_p(\mathbf{X}|\mathbf{Y})|\mathbf{Y}) = f_p(\mathbf{X}|\mathbf{Y})$,
5. (Degradedness) $f_p\left(\mathbf{X}|\mathbf{Y}_{\mathrm{snr}_0}, \mathbf{Y}_{\mathrm{snr}}\right) = f_p\left(\mathbf{X}|\mathbf{Y}_{\mathrm{snr}_0}\right)$, for $\mathbf{X} \to \mathbf{Y}_{\mathrm{snr}_0} \to \mathbf{Y}_{\mathrm{snr}}$,
6. (Shift) $\mathrm{mmpe}(\mathbf{X} + a, \mathrm{snr}, p) = \mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p)$,
7. (Scaling) $\mathrm{mmpe}(a\mathbf{X}, \mathrm{snr}, p) = a^p \mathrm{mmpe}(\mathbf{X}, a^2\mathrm{snr}, p)$.

$$\mathrm{E}\left[f_p\left(X|Y\right)\right] \cancel{=} \mathrm{E}\left[X\right]$$

Bounds on The MMPE

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \min\left(\frac{1}{\mathrm{snr}}, \|\mathbf{X}\|_2^2\right)$$

Theorem. For $\mathrm{snr} \geq 0$, $0 < q \leq p$, and input $\mathbf{X}$

$$\mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p) \leq \min\left(\frac{\|\mathbf{Z}\|_p^p}{\mathrm{snr}^{\frac{p}{2}}}, \|\mathbf{X}\|_p^p\right),$$

$$n^{\frac{p}{q}-1}\mathrm{mme}^{\frac{p}{q}}(\mathbf{X}, \mathrm{snr}, q) \leq \mathrm{mme}(\mathbf{X}, \mathrm{snr}, p) \leq \|\mathbf{X} - \mathrm{E}[\mathbf{X}|\mathbf{Y}]\|_p^p.$$

Bounds on The MMPE

Interpolation Bounds

Theorem. For any $0 < p < q < r \leq \infty$ and $\alpha \in (0,1)$ such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{\bar{\alpha}}{r} \iff \alpha = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}, \tag{1a}$$

where $\bar{\alpha} = 1 - \alpha$,

$$\mathrm{mmpe}^{\frac{1}{q}}(\mathbf{X}, \mathrm{snr}, q) \leq \inf_f \|\mathbf{X} - f(\mathbf{Y})\|_p^{\alpha} \|\mathbf{X} - f(\mathbf{Y})\|_r^{\bar{\alpha}}. \tag{1b}$$

In particular,

$$\mathrm{mmpe}^{\frac{1}{q}}(\mathbf{X}, \mathrm{snr}, q) \leq \|\mathbf{X} - f_r(\mathbf{X}|\mathbf{Y})\|_p^{\alpha} \; \mathrm{mmpe}^{\frac{\bar{\alpha}}{r}}(\mathbf{X}, \mathrm{snr}, r), \tag{1c}$$

$$\mathrm{mmpe}^{\frac{1}{q}}(\mathbf{X}, \mathrm{snr}, q) \leq \mathrm{mmpe}^{\frac{\alpha}{p}}(\mathbf{X}, \mathrm{snr}, p) \|\mathbf{X} - f_p(\mathbf{X}|\mathbf{Y})\|_r^{\bar{\alpha}}. \tag{1d}$$

Bounds on The MMPE

Change of Measure Bound

Theorem. For $0 < \mathrm{snr} \leq \mathrm{snr}_0$, $\mathbf{X}$ and $p \geq 0$, we have

$$\mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p) \leq \kappa_{n,t} \ \mathrm{mmpe}^{\frac{1-t}{1+t}} \left(\mathbf{X}, \mathrm{snr}_0, \frac{1+t}{1-t} \cdot p \right),$$

$$\text{where } \kappa_{n,t} := \left(\frac{2^n}{n^2} \right)^{\frac{t}{t+1}} \left(\frac{1}{1-t} \right)^{\frac{nt}{t+1} - \frac{1}{2}}, \ t = \frac{\mathrm{snr}_0 - \mathrm{snr}}{\mathrm{snr}_0}.$$

Fix SNR But Change Order

New Bound on max-MMSE

Theorem.

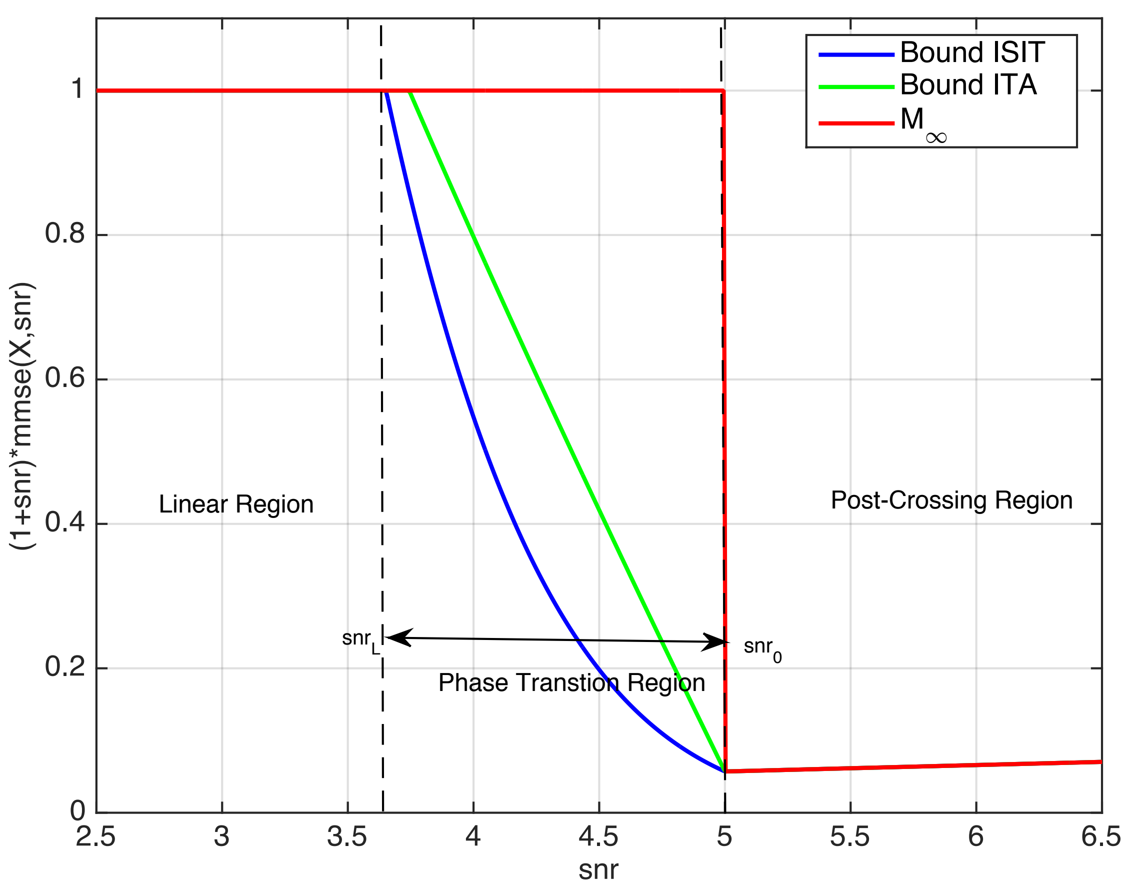
$$T_n(\mathrm{snr}, \mathrm{snr}_0, \beta) \leq \min_{r > \frac{2}{\gamma}} k_{n,\gamma,r} \cdot \mathrm{mmse}^{\frac{\gamma r - 2}{r-2}}(\mathbf{X}, \mathrm{snr}_0),$$

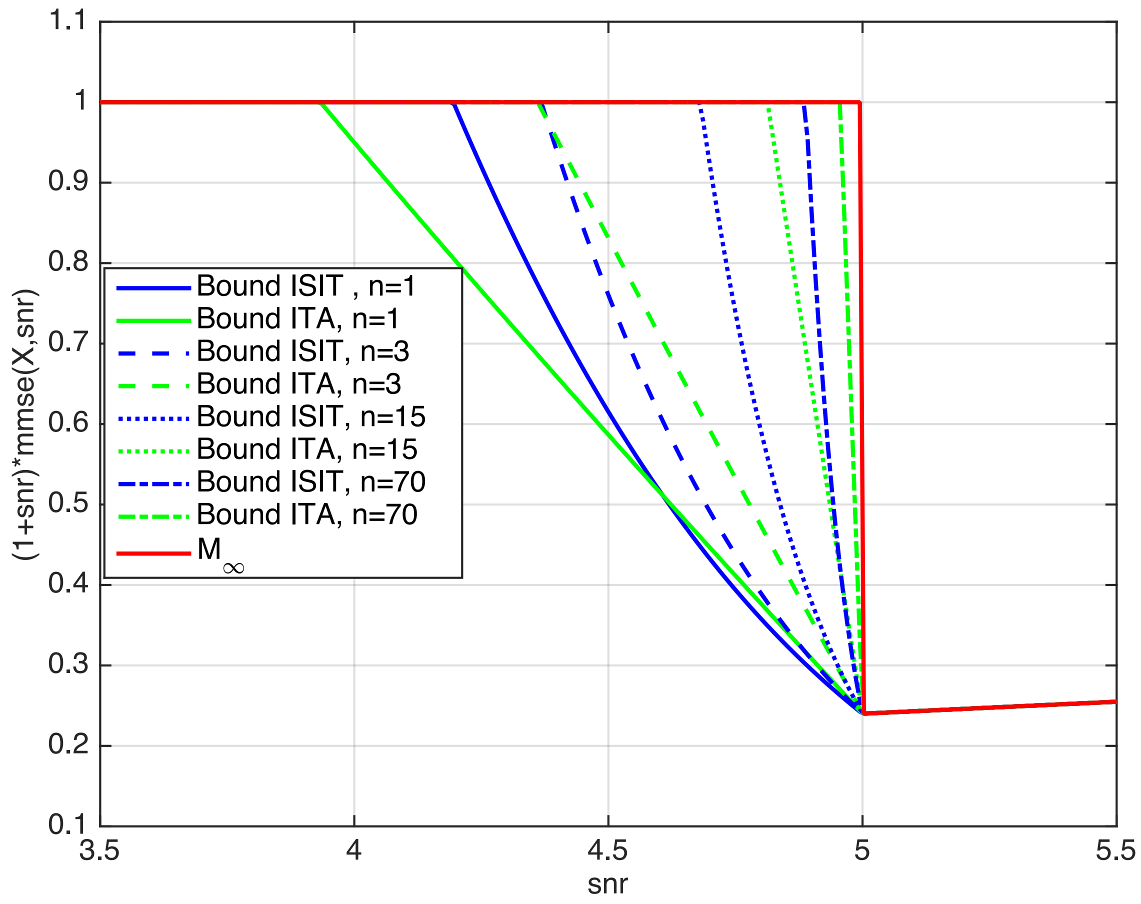
$$\gamma := \frac{\mathrm{snr}}{2\mathrm{snr}_0 - \mathrm{snr}} \in (0, 1],$$

$$k_{n,\gamma,r} := \frac{\sqrt{2}}{n^{1-\gamma}} \left(\frac{1+\gamma}{\gamma}\right)^{\frac{n(1-\gamma)-1}{2}} \left(\frac{2^r \|\mathbf{Z}\|_r^r}{\mathrm{snr}_0^{\frac{r}{2}}}\right)^{\frac{2(1-\gamma)}{r-2}}.$$

Moreover, the width of the phase transition region is given by $W(n) = O\left(n^{-\frac{1}{2}}\right)$.

New Bound on max-MMSE





snr_0=5, beta=0.01, n=1

snr_0=5, beta=0.05

Some New Directions

Bounds on Conditional Entropy

$$h(\mathbf{U}|\mathbf{V}) \le \frac{n}{2} \log(2\pi \mathrm{e}\ \mathrm{mmse}(\mathbf{U}|\mathbf{V})),$$

Theorem. If $\|\mathbf{U}\|_p < \infty$ for some $p \in (0, \infty)$ then for any $\mathbf{V} \in \mathbb{R}^n$, we have

$$h(\mathbf{U}|\mathbf{V}) \le \frac{n}{2} \log\left(k_{n,2p}^2 \cdot n^{\frac{1}{p}} \cdot \mathrm{mmpe}^{\frac{1}{p}}(\mathbf{U}|\mathbf{V};p)\right),$$

where $k_{n,p} := \frac{\sqrt{\pi}\left(\frac{p}{n}\right)^{\frac{1}{p}} \mathrm{e}^{\frac{1}{p}} \Gamma^{\frac{1}{n}}\left(\frac{n}{p}+1\right)}{\Gamma^{\frac{1}{n}}\left(\frac{n}{2}+1\right)} = \sqrt{2\pi\mathrm{e}} \frac{1}{n^{\frac{1}{2}}\left(\frac{p}{2}\right)^{\frac{1}{2n}}} + o\left(\frac{n}{p}\right).$

- Sharper version of Ozarow-Wyner Bound
- Bounds on the Derivative of the MMSE

A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, H. V. Poor, and S. Shamai, “Applications of the Minimum Mean p-th Error (MMPE) to Information Theoretic Quantities,”Accepted to *IEEE Information Theory Workshop,* Cambridge, 2016 https://arxiv.org/pdf/1603.07628, 2016.

Generalized OW Bound

For any $p \geq 1$

$$[H(\mathbf{X}_D) - \mathsf{gap}_p]^+ \leq I(\mathbf{X}_D; \mathbf{Y}) \leq H(\mathbf{X}_D),$$

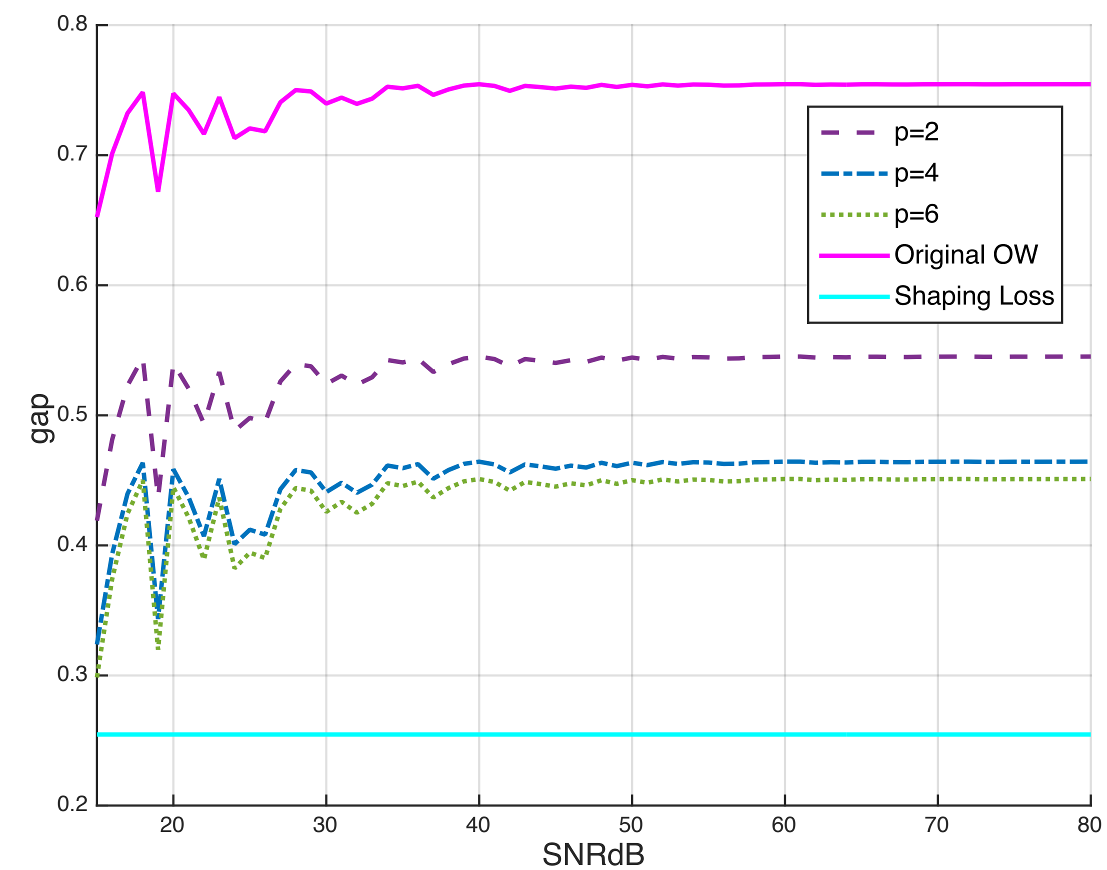
where

$$\mathsf{gap}_p = \inf_{\mathbf{U} \in \mathcal{K}} \mathrm{gap}(\mathbf{U}),$$

$$n^{-1} \cdot \mathrm{gap}(\mathbf{U}) = \log\left(\frac{\|\mathbf{U} + \mathbf{X}_D - f_p(\mathbf{X}_D|\mathbf{Y})\|_p}{\|\mathbf{U}\|_p}\right) + \log\left(\frac{k_{n,p} \cdot n^{\frac{1}{p}} \cdot \|\mathbf{U}\|_p}{\mathrm{e}^{\frac{1}{n} h_{\mathrm{e}}(\mathbf{U})}}\right).$$

where $\mathcal{K} = \{\mathbf{U} : \mathrm{diam}(\mathrm{supp}(\mathbf{U})) \leq 2 \cdot d_{\min}(\mathbf{X}_D)\}$.
Moreover,

$$\log\left(\frac{\|\mathbf{U} + \mathbf{X}_D - f_p(\mathbf{X}_D|\mathbf{Y})\|_p}{\|\mathbf{U}\|_p}\right)$$
$$\overset{\text{for } p \geq 1}{\leq} \log\left(1 + \frac{\mathrm{mmpe}^{\frac{1}{p}}(\mathbf{X}, \mathrm{snr}, p)}{\|\mathbf{U}\|_p}\right).$$



Concluding Remarks

- max-MMSE
- Properties of the MMPE functional
- Bounds on the width of the phase transition region $O(n^{-1/2})$
- Other applications

Thank you

arXiv:1607.01461

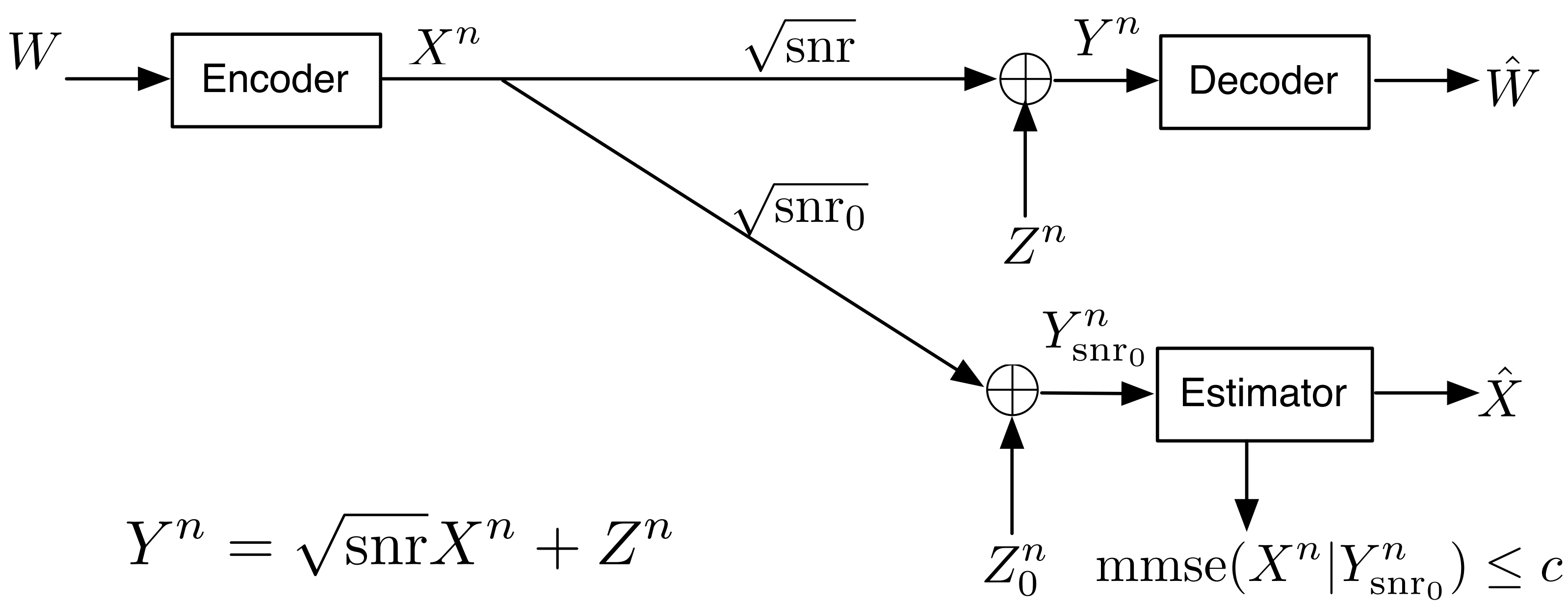
odytso2@uic.edu

Back Up Slides

arXiv:1607.01461

odytso2@uic.edu

Problem



$$
\begin{aligned}
Y^n &= \sqrt{\mathrm{snr}}X^n + Z^n \\
Y^n_{\mathrm{snr}_0} &= \sqrt{\mathrm{snr}_0}X^n + Z_0^n \\
&\text{where } Z^n \sim \mathcal{N}(\mathbf{0}, \mathbf{I}), Z_0^n \sim \mathcal{N}(\mathbf{0}, \mathbf{I})
\end{aligned}
$$

max-I problem

$$
\mathcal{C}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) := \sup_{X^n} \frac{1}{n} I(X^n, Y^n_{\mathrm{snr}}),
$$

$$
\text{s.t.} \operatorname{Tr}\left(\mathbb{E}\left[X^n \left(X^n\right)^{\mathrm{T}}\right]\right) \leq n,
$$

$$
\text{and } \mathrm{mmse}(X^n, \mathrm{snr}_0) \leq \frac{\beta}{1 + \beta \mathrm{snr}_0}.
$$

ISIT 2016 BARCELONA

max-MMSE problem

max-MMSE problem

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) := \sup_{\mathbf{X}} \mathrm{mmse}(\mathbf{X}, \mathrm{snr}),$$
$$\text{s.t. } \frac{1}{n}\mathrm{Tr}\left(\mathbb{E}[\mathbf{X}\mathbf{X}^{\mathrm{T}}]\right) \le 1,$$
$$\text{and } \mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) \le \frac{\beta}{1+\beta \mathrm{snr}_0}.$$

Relationship to max-I problem

$$\frac{\mathrm{snr}}{2}\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) \le \mathcal{C}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) \le \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{M}_n(t, \mathrm{snr}_0, \beta)dt.$$